Comments on "The Depth-Dependent Current and Wave Interaction Equations: A Revision"

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ABSTRACT

Equations for the wave-averaged three-dimensional momentum equations have been published in this journal. It appears that these equations are not consistent with the known depth-integrated momentum balance, especially over a sloping bottom. These equations should thus be considered with caution as they can produce erroneous flows, in particular outside of the surf zone. It is suggested that the inconsistency in the equations may arise from the different averaging operators applied to the different terms of the momentum equation. It is concluded that other forms of the momentum equations, expressed in terms of the quasi-Eulerian velocity, are better suited for three dimensional modelling of wave-current interactions.

1. Introduction

The wave-averaged conservation of momentum can take essentially two forms, one for the mean flow momentum only, and the alternative form for the full momentum, which includes the wave pseudo-momentum (hereinafter 'wave momentum', see McIntyre 1981). This question is well known for depth-integrated equations (Longuet-Higgins and Stewart 1964; Garrett 1976; Smith 2006), but the vertical profiles of the mass and momentum balances are more complex. The pioneering effort of Mellor (2003, hereinafter M03) produced practical wave-averaged for the total momentum that, in principle, may be used in primitive equation models to investigate coastal flows, such as the wave-driven circulations observed by Lentz et al. (2008). The first formulation (Mellor 2003) was slightly inconsistent due to the improper approximation of wave motion with Airy wave theory, which is not enough on a sloping bottom, however small the slope may be. This question was discussed by Ardhuin et al. (2008), and a correction was given and verified. These authors acknowledged that these equations, when using the proper approximation, are not well suited for practical applications because very complex wave models are required for the correct estimation of the vertical fluxes of wave momentum, that are part of the fluxes of total momentum.

Although M03 gave correct wave-forcing expressions – in terms of velocity, pressure and wave-induced displacement, before any approximation – Mellor (2008, hereinafter M08) derived a new and different solution from scratch. The two theories may be consistent over a flat bottom, but they differ at their lowest order over sloping bottoms, so that the M08 equations are likely to be flawed, given the analysis of M03 by Ardhuin et al. (2008), and the fact that their consistency was not verified numerically over sloping bottoms.

Instead, M08 asserted that the equations are consistent with the depth-integrated equations of Phillips (1977). Further, about the test case proposed by Ardhuin et al. (2008), M08 stated that the wave energy was unchanged along the wave propagation and that the resulting wave forcing whould be uniform over the depth. Here we show that the M08 equations do not yield the known depth-integrated equations (Phillips 1977) with a difference that produces very different mean sea level variations when waves propagate over a sloping bottom. As for the test case of proposed by Ardhuin et al. (2008), we show that a consistent analysis should take into account the small but significant change in wave energy due to shoaling. In the absence of dissipative processes, the M08 equations can produce spurious velocities of at least 30 cm/s, with 1 m high waves over a bottom slope of the order of 1% in 4 m water depth.

2. Depth-integration of the M08 equations

For simplicity we consider motions limited to a vertical plane (x, z) with constant water density and no Coriolis force nor wind stress or bottom friction. The wave-averaged momentum equation in M08 takes the form

$$\frac{\partial U}{\partial t}+\frac{\partial U^2}{\partial x}+\frac{\partial UW}{\partial z}=-g\frac{\partial\widehat{\eta}}{\partial x}+F, \tag{1}$$

and the continuity equation is

$$\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}=0. \tag{2}$$

Where U and W are the Lagrangian mean velocity components, which contains the current and Stokes drift velocities, g is the acceleration due to gravity and $\widehat{\eta}$ is the time-averaged water level at the horizontal position x. The force given by M08 on the right hand side of (1) can be written as the sum

$$F=F_{px}^{M08}+F_{uu} \tag{3}$$

of a wave-induced pressure gradient

$$\begin{aligned} F_{px}^{M08} &= -\frac{\partial S_{px}^{M08}}{\partial x}=-\frac{\partial}{\partial x}\left(E_D-\overline{\widetilde{w}^2}\right) && (4)\\ &\simeq -\frac{\partial}{\partial x}\left(E_D-kEF_{SC}F_{SS}\right) && (5)\end{aligned}$$

and the divergences of the horizontal flux of wave momentum,

$$\begin{aligned} F_{uu}^{M08} &= -\frac{\partial S_{uu}}{\partial x}=-\frac{\partial\overline{\widetilde{u}^2}}{\partial x} && (6)\\ &\simeq -\frac{\partial}{\partial x}\left(kEF_{CC}F_{CS}\right) && (7)\end{aligned}$$

where E is the wave energy, k is the wavenumber, $\widetilde{u}$ and $\widetilde{w}$ are respectively the horizontal and vertical wave-induced (orbital) velocities. E_D is defined by

$$E_D=0 \text{ if } z\neq\widehat{\eta} \text{ and } \int_{-h}^{\widehat{\eta}^+}E_D dz=\frac{E}{2}. \tag{8}$$

Using the mean water depth D, and bottom elevation $-h$, F_{CC}, F_{SS} and F_{SC} are non-dimensional functions of kz and kD,

$$\begin{aligned} F_{CC} &= \frac{\cosh(kz+kh)}{\cosh(kD)}, && (9)\\ F_{SS} &= \frac{\sinh(kz+kh)}{\sinh(kD)}, && (10)\\ F_{SC} &= \frac{\sinh(kz+kh)}{\cosh(kD)}. && (11)\end{aligned}$$

The depth-averaged mass-transport velocity is

$$\overline{U}=\frac{1}{D}\int_{-h}^{\widehat{\eta}}U\mathrm{d}z. \tag{12}$$

M08 correctly noted that

$$\int_{-h}^{\widehat{\eta}}\underbrace{\left(S_{px}^{M08}+S_{uu}\right)}_{=S_{xx}^{M08}}\mathrm{d}z=S_{xx}^{P77} \tag{13}$$

with S_{xx}^{P77} given by Phillips (1977). However, for a depth-uniform U, the depth integrated momentum equation in Phillips (1977) is

$$\frac{\partial(D\overline{U})}{\partial t}+\frac{\partial}{\partial x}\left(D\overline{U}^2\right)=-gD\frac{\partial\widehat{\eta}}{\partial x}+\frac{\partial}{\partial x}S_{xx}^{P77}. \tag{14}$$

The forcing in the depth-integration of (1) differs from the forcing in (14), because the gradient is inside of the integral, namely,

$$\begin{aligned}\int_{-h}^{\widehat{\eta}}\frac{\partial S_{xx}^{M08}}{\partial x}dz &= \frac{\partial S_{xx}^{P77}}{\partial x}-S_{xx}^{M08}(z=-h)\frac{\partial h}{\partial x}\\ &\quad -S_{xx}^{M08}(z=\widehat{\eta})\frac{\partial\widehat{\eta}}{\partial x}. && (15)\end{aligned}$$

The depth integral of M08 thus includes two extra term. In particular $S_{xx}^{M08}(z=-h)\partial h/\partial x=-2kE(\partial h/\partial x)/\sinh(2kD)$ can be dominant over a sloping bottom. As a result the momentum balance in M08, unlike M03, does not produce the known set-down and set-up. This is illustrated in figure 1. We take the case proposed by Ardhuin et al. (2008) with steady monochromatic waves shoaling on a slope without breaking nor bottom friction and for an inviscid fluid, conditions in which exact numerical solutions are known. The bottom slopes smoothly from a depth $D=6$ to $D=4$ m. We added a symmetric slope back down to 6 m to allow periodic boundary conditions if needed. For a wave period of 5.24 s the group velocity varies little from 4.89 to 4.64 m s^{-1}, giving a 2.7% increase of wave amplitude on the shoal. Contrary to statements in M08, $\partial E/\partial x$ is significant, with a 5.4% change of E over a few wavelengths.

From the Eulerian analysis of that situation (e.g. Longuet-Higgins 1967), the mean water level should be 0.32 mm lower on the shoal (figure 1). Rivero and Arcilla (1995) established that there is no other dynamical effect: a steady Eulerian mean current develops, compensating for the divergence of the wave-induced mass transport (see also Ardhuin et al. 2008).

3. Flows produced by the M08 equations

Because the relative variation in phase speed is important, from 6.54 to 5.65 m s^{-1}, the Stokes drift accelerates on the shoal. The Eulerian velocity $\widehat{u}$ is irrotational, thus nearly depth-uniform, and compensates the Stokes drift divergence by a convergence. The Lagrangian velocity U, shown in figure 2, is the sum of the two steady velocity fields.

We now solve for the equations derived by M08. The numerical solution is obtained by coupling the WAVEWATCH III wave model (Tolman 2009), solving the phase-averaged wave action equation, and the MARS3D ocean circulation model (Lazure and Dumas 2008). This coupling uses the generic coupler PALM (Buis et al. 2008). The feedback from flow to waves is negligible here and was thus

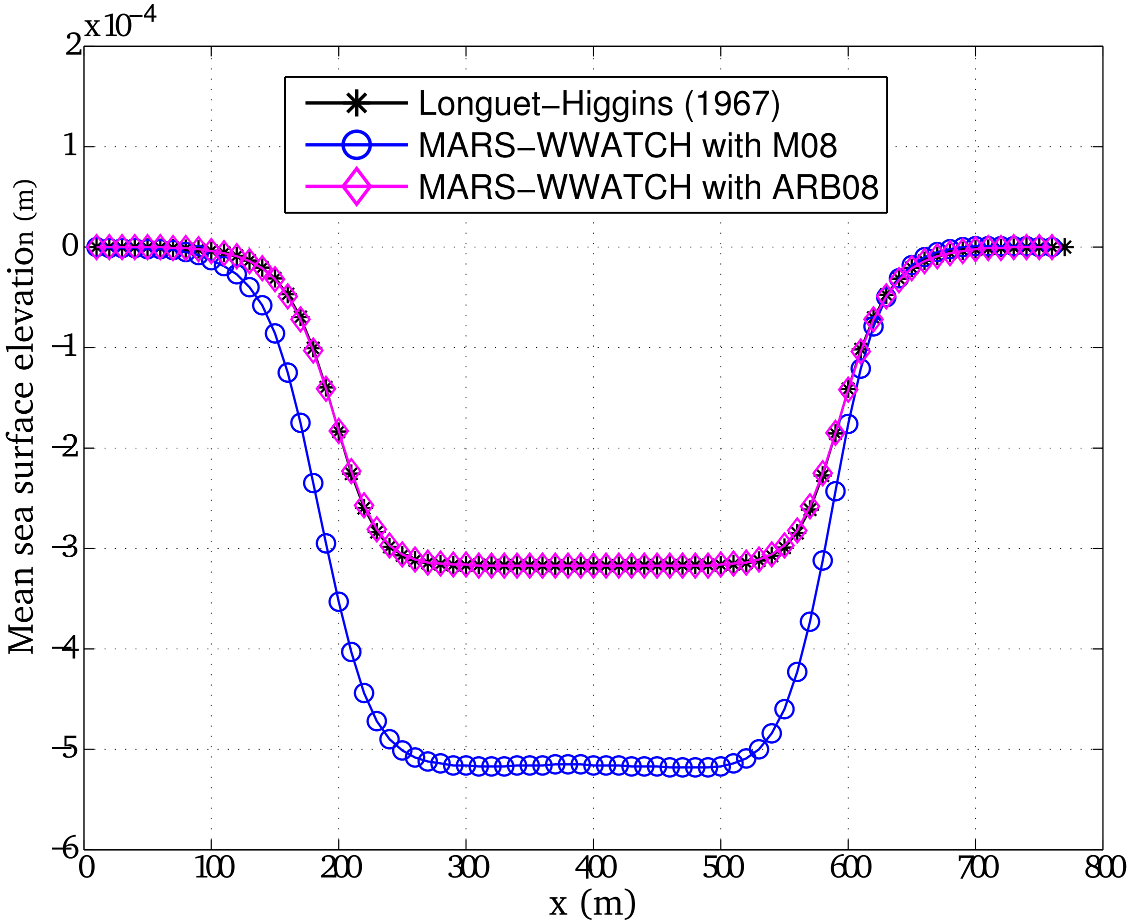

FIG. 1. Mean sea surface elevation induced by monochromatic waves propagating over the smooth bottom shown in figure 2, with amplitude $H_s = 0.34$ m and period $T = 5.24$ s. The extra terms forcing terms in eq. (15) lead to an overestimation of the set-down by more than 50 % for this case. ARB08 stands for quasi-Eulerian momentum equations of Ardhuin et al. (2008).

turned off. MARS3D was implemented with 100 sigma levels regularly spaced, and 5 active points in the transversal y direction, with 2 extra wall points, and 2 ghost points needed to define finite differences, it is thus a reall three-dimensional calculation although the physical situation is two-dimensional. There are 78 active points in the x direction. The time step was set to 0.05 s for tests with $H_s = 1.02$ m (1 s for $H_s = 0.34$ m). For simplicity, the wave model forcing is updated at each time step. We use Eq. (1) transformed to ς coordinates, with ς defined by $z = s(x, \varsigma, t) = \widehat{\eta} + \varsigma D + \widetilde{s}$ (Mellor 2003),

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{W}{D}\frac{\partial U}{\partial \varsigma} = F - g\frac{\partial \widehat{\eta}}{\partial x}. \quad (16)$$

where the advection terms are obtained by using Eq. (2).

The flow boundary conditions are open. The monochromatic wave amplitude $a = 0.12$ m translates into a significant wave height H_s of 0.34 m for random waves with the same energy. We also test the model with $a = 0.36$ m, i.e. $H_s = 1.02$ m, still far from the breaking limit in 4 m depth.

The discontinuity of the vertical profile in the forcing F, due to the E_D term, is not easily ingested by the numerical model, and generates a strongly oscillating velocity profile (Fig. 3). These oscillations are absent at depths larger than 0.8 m, consistent with the zero values of F below the surface. A realistic constant viscosity $K_z = 2.8.10^{-3}$ m^2 s^{-1} removes the oscillations by diffusing the negative term $-\partial E_D/\partial x$ over the vertical. Yet this term is a momentum source that produces velocities one order of magnitude larger than the Stokes drift U_s, with an opposite sign (Fig. 3). The spurious velocities given by M08 with a realistic mixing are most pronounced for waves in not too shallow water (Table 1), and comparable with those given by the M03 equations without mixing.

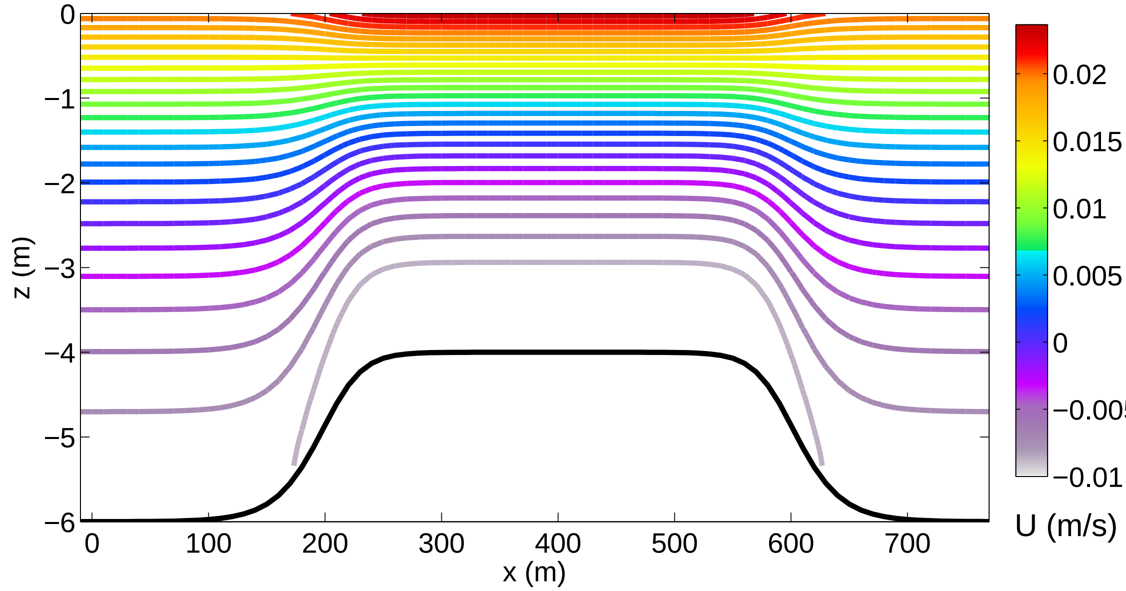

FIG. 2. Lagrangian velocity U for the inviscid sloping bottom case with $H_s = 1.02$ m and $T = 5.24$ s, obtained from the quasi-Eulerian analysis as $U = \widehat{u} + U_s$. Contours are equally spaced from -0.01 to 0.025 m s^{-1}. The thick black line is the bottom elevation.

TABLE 1. **Model results with Mellor (2008):** Surface velocity at x =200 m (on hte up-slope) for different model settings. The settings corresponding to the test in Ardhuin et al. (2008) are given in the second line. The surface velocity values are written for the time $t = 900$ s except for the case without mixing ($t = 360$ s).

H_s(m)	T_p(s)	K_z (m^2 s^{-1})	U (m s^{-1})
1.02	5.6	0	0.6116
0.34	5.6	0	0.2127
0.34	13	0	0.3164
1.02	5.6	$2.8.10^{-3}$	-0.1594
0.34	5.6	$2.8.10^{-3}$	-0.0256
0.34	13	$2.8.10^{-3}$	-0.0007

4. Conclusions

We showed that the equations derived by Mellor (2008) appear inconsistent with the know depth-integrated momentum balances in the presence of a sloping bottom. In the absence of dissipation, a numerical integration of these equation produce unrealistic surface elevations and currents. The currents may reach significant values for very moderate waves, exceeding the expected results by one order of magnitude. While we did not discuss the origin of the inconsistency, it appears that Mellor (2008) used a different averaging for the pressure gradient term and for the advection terms of the same equation. We believe that this is the original reason for the problems discussed here. The spurious velocities produced by M08 are likely to be dwarfed by the strong forcing imposed by breaking waves in the surf zone. Nevertheless, we expect that the M08 equations can produce large errors for continental shelf applications, such as the investigation of cross-shore transports outside of the surf zone. Alternatively, equations for the quasi-Eulerian velocity can be used McWilliams et al. (2004); Ardhuin et al. (2008); Uchiyama et al. (2009) which do not have such problems.

Acknowledgments.

A-C. B. acknowledges the support of a post-doctoral grant from INSU and grant ANR-BLAN-08-0330-01. F.A. is supported by a FP7-ERC grant #240009 “IOWAGA”, and the U.S. National Ocean Partnership Program, under ONR grant N00014-10-1-0383.

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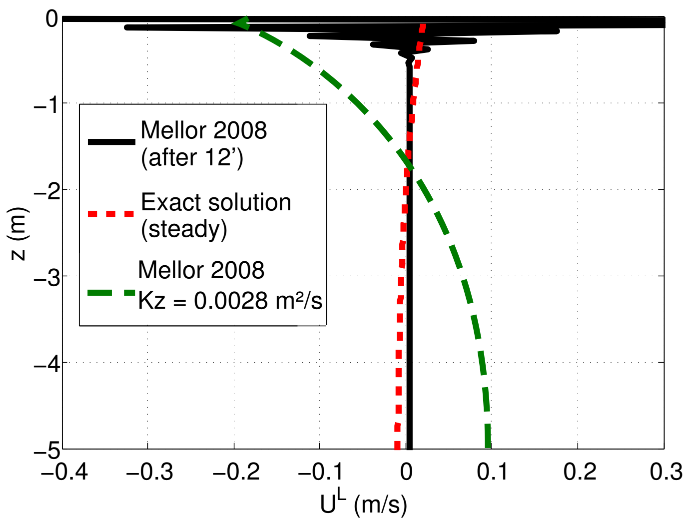

FIG. 3. Comparison of vertical profiles of U at $x = 200$ m given by different models: M08 without mixing (solid black line), M08 with mixing (dashed green line), exact solution (dashed red line). The wave parameters are $H_s = 1.02$ m and $T = 5.26$ s. All profiles are plotted after six minutes of time integration. The x-axis was clipped, and the maximum velocities with M08 reached 0.8 m s^{-1}.

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